

THE COMBINATORIAL NORM OF A MORPHISM OF SCHEMES

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ABSTRACT. In this paper we will prove that there exists a covariant functor from the category of schemes to the category of graphs. This functor provides a combination between algebraic varieties and combinatorial graphs so that the invariants defined on graphs can be introduced to algebraic varieties in a natural manner. By the functor, we will define the combinatorial norm of a morphism of schemes. Then we will obtain some properties of morphisms of norm not great than one. The topics discussed here can be applied to study the discrete Morse theory on arithmetic schemes and Kontsevich's theory of graph homology.

INTRODUCTION

In this paper we will demonstrate a type of combinatorial properties of algebraic varieties from the viewpoint of graph theory.

In §1 we will first prove that there exists a covariant functor Γ , called the graph functor, from the category Sch of schemes to the category $Grph$ of graphs. Here the trick is fortunately built on Weil's specializations^[25].

This functor Γ provides a combination between algebraic varieties and combinatorial graphs (See *Theorem 1.6*). By the graph functor Γ , to each scheme X , assign a combinatorial graph $\Gamma(X)$; to each morphism $f : X \rightarrow Y$ of schemes, assign a homomorphism $\Gamma(f) : \Gamma(X) \rightarrow \Gamma(Y)$ of combinatorial graphs.

For example, it is easily seen that the combinatorial graph $\Gamma(\text{Spec}(\mathbb{Z}))$ of $\text{Spec}(\mathbb{Z})$ is a tree^[9], i.e., a star-shaped graph with the generic point as the center. Thus, the illustrations of $\text{Spec}(\mathbb{Z})$ in [19, 21] are not "correct".

The practical application of the graph functor Γ is that in a natural manner we can exactly introduce into algebraic varieties the invariants that are defined on combinatorial graphs and have been studied in recent decades such as discrete Morse theory^[3–8, 14, 15, 20, 23] and graph homology^[13, 16–18].

However, in general, the combinatorial graphs of most schemes are not finite; some typical schemes rising from arithmetics (for example, see [11]) are of infinite dimensions and hence their graphs are infinite. It follows that

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there is a situation in which it is necessary for one to set up some combinatorial quantity for a morphism of schemes to describe its some property and from its local data to attempt to obtain its global behavior.

Thus, in §2 we will introduce the definition for a combinatorial norm of a morphism of schemes. The norm of a morphism is defined by the graph functor Γ in an evident manner (See *Definition 2.1*). The range of norms of morphisms can be any non-negative integers. For example, a morphism from a scheme to a zero-dimensional scheme has norm zero; an isomorphism of schemes has norm one; a length-preserving morphism has norm one; an injective morphism can have a norm of more than one (See *Example 2.2*, *Remark 2.5*, and *Corollary 2.10*).

We will then conduct an extensive study on some particular types of morphisms of schemes by means of combinatorial graphs and their lengths. As a common characteristic, all these morphisms have the norms of not great than one. In *Proposition 2.6* we will give an application of the length-preserving morphism.

It is easily seen that “injective \nLeftrightarrow length-preserving” and that “norm one \nLeftrightarrow injective” for a morphism of schemes (See *Remark 2.5*). So in *Theorem 2.7* we will give a sufficient condition to a morphism of schemes whose norm is not greater than one. And in *Theorem 2.8* we will obtain a comparison between injective and length-preserving morphisms of schemes.

The results on norms of morphisms between schemes, discussed in the paper, can be applied to topics on the discrete Morse theory on arithmetic schemes and Kontsevich’s theory of graph homology [for example, see our subsequent paper].

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1. THE COMBINATORIAL GRAPH OF A SCHEME

1.1. Notation. Let us recall some definitions in [10,25]. Let E be a topological space E and $x, y \in E$. If y is in the closure $\overline{\{x\}}$, y is a **specialization** of x (or, x is said to be a **generalization** of y) in E , denoted by $x \rightarrow y$. Put $Sp(x) = \{y \in E \mid x \rightarrow y\}$. It is evident that $Sp(x) = \overline{\{x\}}$ is an irreducible closed subset in E .

If $x \rightarrow y$ and $y \rightarrow x$ both hold in E , y is a **generic specialization** of x in E , denoted by $x \leftrightarrow y$. The point x is **generic** (or **initial**) in E if we have $x \leftrightarrow z$ for any $z \in E$ such that $z \rightarrow x$. And x is **closed** (or **final**) if we have $x \leftrightarrow z$ for any $z \in E$ such that $x \rightarrow z$. We say that y is a **closest specialization** of x in X if either $z = x$ or $z = y$ holds for any $z \in X$ such that $x \rightarrow z$ and $z \rightarrow y$.

1.2. Any specialization is contained in an affine open set. Let $E = Spec(A)$ be an affine scheme. For any point $z \in Spec(A)$, denote by j_z

the corresponding prime ideal in A . It is clear that there is a specialization $x \rightarrow y$ in $\text{Spec}(A)$ if and only if $j_x \subseteq j_y$ in A . It follows that there is a generic specialization $x \leftrightarrow y$ in $\text{Spec}(A)$ if and only if $x = y$. Now given a scheme X .

Lemma 1.1. *Let $x, y \in X$. Then we have $x \leftrightarrow y$ in X if and only if $x = y$.*

Proof. \Leftarrow . Trivial. Prove \Rightarrow . Assume $x \leftrightarrow y$ in X . Let U be an affine open set of X containing x . From $x \leftrightarrow y$ in X , we have $Sp(x) = Sp(y)$; then $x \in Sp(x) \cap U = Sp(y) \cap U \ni y$, that is, y is contained in U . Hence, $x \leftrightarrow y$ in U . It follows that $x = y$ holds in U (and of course in X). \square

Lemma 1.2. *Any specialization $x \rightarrow y$ in X is contained in some affine open subset U of X , that is, the points x, y are both contained in U . In particular, each affine open set of X containing y must contain x .*

Proof. Take a specialization $x \rightarrow y$ in X with $x \neq y$. Then y is a limit point of the one-point set $\{x\}$ since y is contained in the topological closure $Sp(x)$ of $\{x\}$. Let $U \subseteq X$ be an open set containing y . We have $U \cap (\{x\} \setminus \{y\}) \neq \emptyset$ by the definition for a limit point of a set (see any standard textbook for general topology). We choose U to be an affine open set of X . \square

1.3. Any morphism preserves specializations. Let E be a topological space and let $IP(W)$ be the set of the generic points in a subset W of E .

E is said to be of the (UIP) –**property** if E satisfies the conditions:

(i) $IP(W)$ is a nonvoid set for any nonvoid irreducible closed subset W of E ; (ii) for any irreducible closed subset V and W of E with $V \neq W$, there is $x_V \neq x_W$ for any $x_V \in IP(V)$ and any $x_W \in IP(W)$.

Let $f : E \rightarrow F$ be a map of spaces. Then f is said to be IP –**preserving** if we have $f(x_0) \in IP(\overline{f(U)})$ for any closed subset U of E and any $x_0 \in IP(U)$. Here $\overline{f(U)}$ denotes the topological closure of the set $f(U)$.

The map f is said to be **specialization-preserving** if there is a specialization $f(x) \rightarrow f(y)$ in F for any specialization $x \rightarrow y$ in E .

Remark 1.3. By Zorn's Lemma it is seen that any irreducible T_0 –spaces have the (UIP) –property if there are generic points. In particular, any scheme is of the (UIP) –property.

Proposition 1.4. Let $f : E \rightarrow F$ be a continuous map of topological spaces.

(i) f is specialization-preserving if and only if f is IP –preserving.

(ii) Let F be of the (UIP) –property. Then f is specialization-preserving.

Proof. (i) \Rightarrow . Let f be specialization-preserving. Take any closed subset U of E and any $x_0 \in IP(U)$. Without loss of generality, we assume that U is irreducible.

For any $x \in U$, there is a specialization $x_0 \rightarrow x$ in U . From the assumption we have a specialization $f(x_0) \rightarrow f(x)$ in $f(U)$; then $f(x_0) \rightarrow y$ in $f(U)$ holds for any $y \in f(U)$. Put

$$Sp(f(x_0))|_{f(U)} = \{z \in f(U) \mid f(x_0) \rightarrow z \text{ in } f(U)\}.$$

As $Sp(f(x_0)) \mid_{f(U)} = f(U)$, we have

$$Sp(f(x_0)) = \overline{Sp(f(x_0)) \mid_{f(U)}} = \overline{f(U)}.$$

It follows that for any $z \in \overline{f(U)}$ there is a specialization $f(x_0) \rightarrow z$ in $\overline{f(U)}$. Hence,

$$f(x_0) \in IP(\overline{f(U)}).$$

\Leftarrow . Let f be IP-preserving. Take any specialization $x_0 \rightarrow x$ in E . Let $U = Sp(x_0)$. We have $f(x_0), f(x) \in f(U)$; then

$$f(x) \in \overline{f(U)} = Sp(f(x_0));$$

hence there is a specialization $f(x_0) \rightarrow f(x)$ in F .

(ii) Fixed any specialization $x \rightarrow y$ in E . From the irreducibility of $Sp(x)$, it is seen that $\overline{f(Sp(x))}$ is an irreducible closed subset in F . As $f(x) \in f(Sp(x))$, there is

$$Sp(f(x)) \subseteq \overline{f(Sp(x))};$$

as F has the (UIP)–property, it is seen that

$$\overline{f(Sp(x))} = Sp(f(x))$$

holds since they both contain $f(x)$ as a generic point. Similarly, we have

$$\overline{f(Sp(y))} = Sp(f(y)).$$

As $Sp(x) \supseteq Sp(y)$, we have $f(Sp(x)) \supseteq f(Sp(y))$; then $Sp(f(x)) \supseteq Sp(f(y))$. So there is a specialization $f(x) \rightarrow f(y)$ in F . \square

For the case of schemes, we have the following lemma.

Lemma 1.5. *Any morphism of schemes is specialization-preserving.*

Proof. It is immediate from Remark 1.3 and Lemma 1.4. \square

1.4. The graph functor Γ from schemes to graphs. Now we have such a covariant functor from the category of schemes to the category of graphs.

Theorem 1.6. *There exists a covariant functor Γ from the category Sch of schemes to the category $Grph$ of graphs defined in such a natural manner:*

(i) *To each scheme X , assign the graph $\Gamma(X)$ in which the vertex set is the set of points in the underlying space X and the edge set is the set of specializations in X .*

Here, for any points $x, y \in X$, we say that there is an edge from x to y if and only if there is a specialization $x \rightarrow y$ in X .

(ii) *To each morphism $f : X \rightarrow Y$ of schemes, assign the homomorphism $\Gamma(f) : \Gamma(X) \rightarrow \Gamma(Y)$ of graphs such that any specialization $x \rightarrow y$ in the scheme X as an edge in $\Gamma(X)$, is mapped by $\Gamma(f)$ into the specialization $f(x) \rightarrow f(y)$ as an edge in $\Gamma(Y)$.*

Proof. It is immediate from Lemmas 1.1 and 1.5. \square

The above functor Γ from the category Sch of schemes to the category $Grph$ of graphs is said to be the **graph functor** in the paper. For a scheme X , the graph $\Gamma(X)$ is said to be the **(associated) graph** of X ; for a morphism $f : X \rightarrow Y$ of schemes, the graph homomorphism $\Gamma(f)$ is said to be the **(associated) homomorphism** of f .

In general, the graph $\Gamma(X)$ of a scheme X is not a finite graph. For example, the graph $\Gamma(\text{Spec}(\mathbb{Z}))$ of $\text{Spec}(\mathbb{Z})$ is a star-shaped graph. The graph $\Gamma(\text{Spec}(\mathbb{Z}[t]))$ of $\text{Spec}(\mathbb{Z}[t])$ is a graph of infinitely many loops.

Remark 1.7. By the graph functor Γ , many invariants defined on graphs can be introduced into algebraic varieties in a natural manner, such as the discrete Morse theory^[3–8,15,23] and the Kontsevich’s graph homology^[13,16–18].

The graph functor Γ can provide us some type of completions of birational maps between algebraic varieties.

2. THE COMBINATORIAL NORM OF A MORPHISM

In this section the graph functor Γ will be applied to set up a combinatorial quantity for a morphism of schemes, called the norm of a morphism. For convenience, in the following we will identify a scheme X with its graph $\Gamma(X)$ and identify a specialization X with its edge in $\Gamma(X)$.

Notice that here a scheme is not necessarily finite-dimensional except when otherwise specified.

2.1. Definition and notation. By the graph functor Γ we have the notion of combinatorial quantities in a scheme which we borrow from graph theory. Let X be a scheme. Fixed a specialization $x \rightarrow y$ in X (regarded as an edge in the graph $\Gamma(X)$).

By a **restrict chain of specializations** $\Delta(x, y)$ (of **length** n) from x to y in X , we understand a chain of specializations

$$x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n = y$$

in X , where $x_i \neq x_{i+1}$ and each x_{i+1} is a specialization of x_i for $0 \leq i \leq n-1$.

The **length** $l(x, y)$ of the specialization $x \rightarrow y$ is the supremum among all the lengths of restrict chain of specializations from x to y . Let W be a subset of X . The **length** $l(W)$ of W is defined to be

$$\sup\{l(x, y) \mid \text{there is a specialization } x \rightarrow y \text{ in } W\}.$$

In particular, the **length** $l(x)$ of a point $x \in X$ is defined to be the length of the subspace $Sp(x)$ in X .

Let $l(W) < \infty$. A restrict chain Δ of specializations in W is a **presentation** for the length $l(W)$ of W if the length of Δ is equal to $l(W)$.

Let $\dim X < \infty$. We have $l(X) = \dim X$. Moreover, let $\Delta(x_0, x_n)$ be a presentation for the length of X . Then x_0 is generic and x_n is closed in X .

2.2. The norm of a morphism between schemes. By the graph functor Γ we can define the combinatorial norm of a morphism between schemes.

Definition 2.1. Let $f : X \rightarrow Y$ be a morphism of schemes.

(i) f is said to be **bounded** if there exists a constant $\beta \in \mathbb{R}$ such that

$$l(f(x_1), f(x_2)) \leq \beta \cdot l(x_1, x_2)$$

holds for any specialization $x_1 \rightarrow x_2$ in X with $l(x_1, x_2) < \infty$.

(ii) Let f be bounded. If $\dim X = 0$, define $\|f\| = 0$; if $\dim X > 0$, define

$$\|f\| = \sup\left\{\frac{l(f(x_1), f(x_2))}{l(x_1, x_2)} : x_1 \rightarrow x_2 \text{ in } X, 0 < l(x_1, x_2) < \infty\right\}.$$

Then the number $\|f\|$ is said to be the **norm** of f .

Example 2.2. The norm of a morphism of schemes can be equal to any non-negative integer.

(i) The k -rational points of a k -variety are morphisms of norm zero.

(ii) Let s, t be variables over a field k and let

$$f : \operatorname{Spec}(k[s, t]) \rightarrow \operatorname{Spec}(k[t])$$

be induced from the evident embedding of k -algebras. Then $\|f\| = 1$.

(iii) Let t be a variable over \mathbb{Q} and let

$$f : \operatorname{Spec}(k[t]) \rightarrow \operatorname{Spec}(\mathbb{Z}[t])$$

be induced from the evident embedding. Then $\|f\| = 2$.

Take a scheme X . Two points x and y in X are said to be **Sp -connected** if either $x \rightarrow y$ or $y \rightarrow x$ holds in X . Otherwise, x and y are said to be **Sp -disconnected** if they are not Sp -connected.

A nonvoid subset A in X is said to be **Sp -connected** if any two elements in A are Sp -connected.

By the norm of a morphism and the graph functor Γ it is seen that there are two specified types of data, the latitudinal data and the longitudinal data, for us to describe a morphisms of scheme (such as *Remarks 2.3-4*).

Remark 2.3. (The Latitudinal Data). Let $f : X \rightarrow Y$ be a morphism of schemes. There are some cases for the latitudinal data such as the following for one to describe f :

(i) f is said to be **level-separated** if the points $f(x)$ and $f(y)$ are Sp -disconnected in Y for any $x, y \in X$ that are Sp -disconnected and of the same lengths.

(ii) f is said to be **level-reduced** if $f(x)$ and $f(y)$ are Sp -connected in Y for any $x, y \in X$ that are Sp -disconnected and of the same lengths.

(iii) f is said to be **level-mixed** if f is neither level-separated nor level-reduced.

Remark 2.4. (The Longitudinal Data). Let $f : X \rightarrow Y$ be a morphism of schemes. There are some cases for the longitudinal data such as the following for one to describe f :

- (i) f is said to be **null** if $\|f\| = 0$.
- (ii) f is said to be **asymptotic** if $\|f\| = 1$.
- (iii) f is said to be **length-preserving** if $l(f(x), f(y)) = h$ holds for any specialization $x \rightarrow y$ in X such that $l(x, y) = h < \infty$.

Remark 2.5. Let $f : X \rightarrow Y$ be a morphism of schemes.

- (i) Let $1 \leq \dim X = \dim Y < \infty$. It is seen that $\|f\| \geq 1$ if f is surjective.
- (ii) Let f be length-preserving. Then $\dim X \leq \dim Y$ and $\|f\| = 1$ hold. In general, it is not true that f is injective. Conversely, let f be injective. In general, it is not true that f is length-preserving.
- (iii) Let $\|f\| = 1$. In general, it is not true that f is injective. Conversely, let f be injective. In general, it is not true that $\|f\| = 1$ holds (See *Corollary 2.11*).

In the following there will be an extensive study on several particular morphisms between schemes by means of combinatorial graphs and their lengths. As a common characteristic, all these morphisms have the norms of not great than one.

2.3. An application of a length-preserving morphism. A morphism $f : X \rightarrow Y$ of schemes is said to be of **finite J -type** if f is of finite type and the homomorphism

$$f^\#|_V : \mathcal{O}_Y(V) \rightarrow f_*\mathcal{O}_X(U)$$

of rings is of J -type for any affine open sets V of Y and U of $f^{-1}(V)$.

Here, a homomorphism $\tau : R \rightarrow S$ of commutative rings is said to be of **J -type** if there is an identity

$$\tau^{-1}(\tau(I)S) = I$$

for every prime ideal I in R .

Proposition 2.6. Let $f : X \rightarrow Y$ be a morphism between irreducible schemes. Suppose $\dim X < \infty$. Then we have

$$\dim X = \dim Y < \infty$$

if f is length-preserving and of finite J -type.

Proof. Let f be length-preserving and of finite J -type. We have

$$l(X) = l(f(X)) \leq l(Y).$$

It is seen that $\dim X \leq \dim Y$ holds since $\dim X = l(X)$ and $(Y) = \dim Y$ hold.

Let $x \in X$ and $y = f(x) \in Y$. As f is of finite J -type, there are affine open subsets V of Y and U of $f^{-1}(V)$ such that

$$f^\#|_V : \mathcal{O}_Y(V) \rightarrow f_*\mathcal{O}_X(U)$$

is a homomorphism of J -type.

Let $V = \text{Spec}(R)$ and $U = \text{Spec}(S)$. We have $\dim U = \dim X$ and $\dim V = \dim Y$.

Take any restrict chain of specializations

$$y_0 \rightarrow y_1 \rightarrow \cdots \rightarrow y_n$$

in V . By §1.2 we obtain a chain of prime ideals

$$j_{y_0} \subsetneq j_{y_1} \subsetneq \cdots \subsetneq j_{y_n}$$

in R , where each j_{y_i} denotes the prime ideal in R corresponding to the point y_i in V .

By *Corollary 2.3*^[22] it is seen that there are a chain of prime ideals

$$I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n$$

in S such that

$$f^{\#-1}(I_i) = j_{y_i}.$$

It follows that there is a restrict chain of specializations

$$x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n$$

in U such that $f(x_i) = y_i$ and $j_{x_i} = I_i$. Hence, $l(U) \geq l(V)$ holds.

It is evident that

$$l(U) = \dim U \text{ and } l(V) = \dim V$$

hold for the subspaces U, V . So we must have

$$\infty > \dim X \geq \dim Y.$$

This completes the proof. \square

2.4. A sufficient condition to a morphism of norm not greater than one. Let $f : X \rightarrow Y$ be a morphism of schemes. Fixed a point $x_0 \in X$. Then f is said to be **Sp-connected** at x_0 if the pre-image $f^{-1}(Sp(f(x_0)))$ is a Sp -connected set. And f is said to be **Sp-proper** at x_0 if the pre-image $f^{-1}(Sp(f(x_0)))$ is equal to $Sp(x_0)$.

The morphism f is said to be of **Sp-type** on X if f is either Sp -connected or Sp -proper at each point $x \in X$.

In fact, such a datum locally defined by specializations can control the global behavior of a morphism of schemes^[11]. For example, a morphism induced by a homomorphism of Dedekind domains of schemes is of Sp -type. An isomorphism of schemes is of Sp -type; the converse is not true.

Here we give a sufficient condition to a morphism of norm not greater than one.

Theorem 2.7. *Let $f : X \rightarrow Y$ be a morphism of schemes. Then we have*

$$0 \leq \|f\| \leq 1$$

if f is of Sp -type on X .

Proof. If $\dim X = 0$ or $\dim Y = 0$ we have $\|f\| = 0$. In the following we assume $\dim X > 0$ and $\dim Y > 0$.

Fixed any specialization $x_1 \rightarrow x_2$ in X such that

$$0 < (x_1, x_2) < \infty$$

and

$$l(f(x_1), f(x_2)) > 0.$$

We will proceed in two steps.

Step 1. Let $x_1 \rightarrow x_2$ in X be a closest specialization. In the following we will prove that the specialization $f(x_1) \rightarrow f(x_2)$ in Y is also closest.

Hypothesize that the specialization $f(x_1) \rightarrow f(x_2)$ in Y is not a closest one. It follows that for the length we have

$$l(f(x_1), f(x_2)) \geq 2.$$

Then

$$\begin{aligned} Sp(f(x_1)) &\supsetneq Sp(f(x_2)); \\ l(Sp(f(x_1))) &= \dim Sp(f(x_1)) \geq 2. \end{aligned}$$

Take a point $y_0 \in Y$ such that

$$f(x_1) \neq y_0 \neq f(x_2)$$

and that there are specializations

$$f(x_1) \rightarrow y_0 \rightarrow f(x_2)$$

in Y . We have

$$Sp(f(x_1)) \supsetneq Sp(y_0) \supsetneq Sp(f(x_2)).$$

As f is of Sp-type, it is seen that there are two cases for the point x_1 .

Case (i). Let f be Sp-proper at x_1 . That is, $Sp(x_1) = f^{-1}(Sp(f(x_1)))$.

Then $f(Sp(x_1)) = Sp(f(x_1))$ holds. As $y_0 \in Sp(f(x_1))$, it is seen that there is a point $x_0 \in Sp(x_1)$ such that $y_0 = f(x_0)$. Hence, we obtain a specialization $x_1 \rightarrow x_0$ in X .

Similarly, there are two subcases for the point x_0 such as the following:

Subcase (i_a). Assume $Sp(x_0) = f^{-1}(Sp(f(x_0)))$.

As $f(x_2) \in Sp(f(x_0))$, it is seen that the point x_2 is contained in the set $Sp(x_0)$; then we have a specialization $x_0 \rightarrow x_2$ in X . So there are specializations

$$x_1 \rightarrow x_0 \rightarrow x_2$$

in X .

Subcase (i_b). Assume that $f^{-1}(Sp(f(x_0)))$ is a Sp-connected set.

Then either $x_0 \rightarrow x_2$ or $x_2 \rightarrow x_0$ is a specialization in X ; by Proposition 2.5 it is seen that only $x_0 \rightarrow x_2$ holds in X since $y_0 \neq f(x_2)$ and $y_0 \rightarrow f(x_2)$; then there are specializations

$$x_1 \rightarrow x_0 \rightarrow x_2$$

in X .

Case (ii). Let f be Sp -connected at x_1 . That is, $f^{-1}(\text{Sp}(f(x_1)))$ is a Sp -connected set.

As $y_0 \in \text{Sp}(f(x_1))$, we have $x_0 \in \text{Sp}(x_1)$ with $y_0 = f(x_0)$. As $y_0 \neq f(x_1)$, it is seen that there is a specialization $x_1 \rightarrow x_0$ in X . As $x_2 \in \text{Sp}(x_1)$ and $y_0 \neq f(x_2)$, we have a specialization $x_0 \rightarrow x_2$ in X ; then we obtain specializations

$$x_1 \rightarrow x_0 \rightarrow x_2.$$

From the above cases, we must have $l(x_1, x_2) \geq 2$. Hence, $x_1 \rightarrow x_2$ in X is not a closest specialization, which is in contradiction to the assumption. Therefore, $f(x_1) \rightarrow f(x_2)$ must be a closest specialization in Y .

Step 2. Let $x_1 \rightarrow x_2$ in X be not closest. Put $l(x_1, x_2) = n \geq 2$. There are the closest specializations

$$z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_{n+1}$$

in X with $z_1 = x_1$ and $z_{n+1} = x_2$.

It is seen that either for some $1 \leq i \leq n$ there is an identity

$$f(z_i) = f(z_{i+1})$$

or

$$f(z_1) \rightarrow f(z_2) \rightarrow \cdots \rightarrow f(z_{n+1})$$

are a restrict chain of specializations in Y . By *Step 1* we have

$$l(f(x_1), f(x_2)) \leq l(x_1, x_2).$$

This proves $\|f\| \leq 1$. □

2.5. A theorem on the comparison between injective and length-preserving morphisms of schemes. A scheme X is said to be **caténaire** if the underlying space of X is a caténaire space. For caténaire spaces, see *ch 8, §1 of [1]*.

Let $x \rightarrow y \rightarrow z$ be specializations in a caténaire scheme. It is clear that

$$l(x, z) = l(x, y) + l(y, z)$$

holds by definition for caténaire space^[1].

Now we obtain a result on the comparison between injective and length-preserving morphisms of schemes.

Theorem 2.8. *Let $f : X \rightarrow Y$ be a morphism of irreducible schemes. Suppose $\dim Y < \infty$.*

Let f be injective and of Sp -type. Then f is length-preserving and level-separated.

Conversely, let X be caténaire. Then f is injective if f is length-preserving and level-separated.

Proof. (i). Prove the first half of the theorem. Let f be injective and of Sp -type. We will prove that f is length-preserving and level-separated.

It is seen that $\dim X < \infty$ holds. Otherwise, hypothesize $\dim X = \infty$. For any $n \in \mathbb{N}$ we have a chain of irreducible closed subsets

$$X_n \subsetneq X_{n-1} \subsetneq \cdots \subsetneq X_0$$

in X . By *Remark 1.3*, there are points $v_j \in X_j$ such that $Sp(v_j) = X_j$ for $0 \leq j \leq n$. Then we have a chain of specializations

$$v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_n$$

in X . By *Lemma 1.5* it is seen that there are specializations

$$f(v_0) \rightarrow f(v_1) \rightarrow \cdots \rightarrow f(v_n)$$

in Y . Hence, $n \leq l(Y) = \dim(Y)$, where we will obtain a contradiction.

As $\dim X < \infty$, we have $l(X) = \dim X$. In the following we will proceed in three steps.

Step 1. Show f is length-preserving. Take a chain of specializations

$$z_0 \rightarrow z_1 \rightarrow \cdots \rightarrow z_n$$

in X such that $l(z_0, z_n) = n$. We have specializations

$$f(z_0) \rightarrow f(z_1) \rightarrow \cdots \rightarrow f(z_n)$$

in Y .

As f is of Sp-type, by *Theorem 2.7* it is seen that $\|f\| \leq 1$ and then $l(f(z_0), f(z_n)) \leq l(z_0, z_n) = n$; as f is injective, it is seen that $f(z_i) \neq f(z_j)$ holds for all $i \neq j$; hence we have

$$l(z_0, z_n) = l(f(z_0), f(z_n)) = n.$$

It follows that

$$l(x, y) = l(f(x), f(y))$$

holds for any specialization $x \rightarrow y$ of finite length. This proves that f is length-preserving.

Step 2. Show that $l(f(z)) = l(z) < \infty$ holds for any $z \in X$.

In fact, by *Step 1* above we have $l(z) \leq l(X) = \dim X < \infty$ for any $z \in X$. As f is length-preserving, we have $l(z) \leq l(f(z))$ by choosing a presentation of specializations for the length of the subspace $Sp(z)$.

To prove $l(z) \geq l(f(z))$, we have two cases for the point z from the assumption that f is of Sp-type.

Case (i). Let $Sp(z) = f^{-1}(Sp(f(z)))$.

As $l(f(z)) \leq l(Y) = \dim Y < \infty$, it is easily seen that there exists some point $w \in Sp(f(z))$ such that

$$l(f(z), w) = l(f(z))$$

by taking a presentation of specializations for the length of the subspace $Sp(f(z))$. Take a point $u \in Sp(z)$ such that $w = f(u)$. As we have proved in *Step 1* that f is length-preserving, we obtain

$$l(f(z)) = l(f(z), w) = l(z, u) \leq l(z).$$

Case (ii). Let $f^{-1}(Sp(f(z)))$ be Sp -connected.

Prove $l(f(z)) \leq l(z)$. In deed, if $l(f(z)) = 0$, we have

$$l(f(z)) = 0 \leq l(z).$$

Let $l(f(z)) \geq 1$. Hypothesize that there is some point $v \in Sp(f(z))$ such that

$$l(f(z)) \geq l(f(z), v) \geq 1 + l(z).$$

Take a point $x \in f^{-1}(Sp(f(z)))$ with $f(x) = v$. As the points x and z are both contained in $f^{-1}(Sp(f(z)))$, it is seen that either $z \rightarrow x$ or $x \rightarrow z$ is a specialization from the assumption above.

If $z \rightarrow x$ is a specialization, we have $l(f(z), v) = l(z, x)$ since f is length-preserving; then

$$1 + l(z) \leq l(f(z), f(x)) = l(z, x) \leq l(z),$$

where there will be a contradiction.

If $x \rightarrow z$ is a specialization, we have a specialization $f(x) \rightarrow f(z)$ by *Lemma 1.5*; as the point $f(x) = v$ is contained in the set $Sp(f(z))$, there is a generic specialization $f(x) \leftrightarrow f(z)$. By *Lemma 1.1* it is seen that $v = f(z)$ holds. Then we have

$$0 = l(v, v) = l(f(z), v) \geq 1 + l(z) \geq 1,$$

where we will obtain a contradiction.

Hence, we must have $l(f(z)) \leq l(z)$. This proves $l(f(z)) = l(z)$ for any $z \in X$.

Step 3. Show f is level-separated. Let $x_1, x_2 \in X$ be Sp -disconnected with $l(x_1) = l(x_2)$. We have $x_1 \neq x_2$ by *Lemma 1.1*.

Then $f(x_1)$ and $f(x_2)$ are Sp -disconnected. Otherwise, hypothesize that there is a specialization $f(x_1) \rightarrow f(x_2)$ in Y . Consider the irreducible closed subsets

$$Sp(f(x_1)) \supseteq Sp(f(x_2)).$$

By *Step 2* we have

$$l(f(x_1)) = l(x_1) = l(x_2) = l(f(x_2));$$

then

$$\dim Sp(f(x_1)) = l(f(x_1)) = l(f(x_2)) = \dim Sp(f(x_2)) < \infty.$$

It follows that

$$Sp(f(x_1)) = Sp(f(x_2))$$

holds.

By *Remark 1.3* we have $f(x_1) = f(x_2)$ as generic points of the irreducible closed set. As f is injective, we must have $x_1 = x_2$, where there will be a contradiction to the assumption above. This proves that $f(x_1)$ and $f(x_2)$ are Sp -disconnected.

(ii). Prove the other half of the theorem. Let X be caténaire and let f be length-preserving and level-separated. We will prove that f is injective.

It is seen that $\dim X = l(X) < \infty$. In deed, if $\dim X = \infty$, we have $l(X) = \infty$ and then $\dim Y \geq l(Y) = \infty$ since f is length-preserving, which is in contradiction to the assumption.

Now fixed any $x, y \in X$. Let ξ be the generic point of X . In the following we will prove $f(x) \neq f(y)$ if $x \neq y$.

There are three cases such as the following.

Case (i). Let $\dim X = 0$.

We have $x = y$ and of course f is injective.

Case (ii). Let $\dim X > 0$ and let $x = \xi$ without loss of generality.

If $x \neq y$, we have $y \neq \xi$ and then $x \rightarrow y$ is a specialization in X . It follows that $l(x, y) > 0$ holds. As f is length-preserving, We have

$$l(f(x), f(y)) = l(x, y) > 0.$$

Hence, $f(x) \neq f(y)$.

Case (iii). Let $\dim X > 0$, $x \neq \xi$ and $y \neq \xi$.

As $l(X) = \dim X < \infty$, for any $z \in X$ we have

$$l(z) \leq l(X) < \infty.$$

There are several subcases such as the following.

Subcase (iii_a). Let $l(x) = l(y)$ and let x, y be Sp-connected.

Assume $y \in Sp(x)$ without loss of generality. We have

$$Sp(x) \supsetneq Sp(y);$$

$$\dim(Sp(x)) = l(x) = l(y) = \dim(Sp(y)).$$

Then $Sp(x) = Sp(y)$ and hence $x = y$. So we have $f(x) = f(y)$. Such a subcase is trivial.

Subcase (iii_b). Let $l(x) = l(y)$ and let x, y be Sp-disconnected.

We have $x \neq y$. As f is level-separated, we have $f(y) \notin Sp(f(x))$; hence,

$$f(x) \neq f(y).$$

Subcase (iii_c). Let $l(x) > l(y)$ without loss of generality and let x, y be Sp-connected.

We have $x \neq y$. It is clear that only $x \rightarrow y$ is a specialization. As f is length-preserving, We have

$$l(f(x), f(y)) = l(x, y) > 0.$$

Hence,

$$f(x) \neq f(y).$$

Subcase (iii_d). Let $l(x) > l(y)$ without loss of generality and let x, y be Sp-disconnected.

We have $x \neq y$. Prove $f(x) \neq f(y)$.

In fact, choose a presentation $\Gamma(x, u)$ of specializations

$$x \rightarrow \cdots \rightarrow x_0 \rightarrow \cdots \rightarrow u$$

in X for the length $l(x) < \infty$. That is, $l(x, u) = l(x)$. Here we have some point $x_0 \in Sp(x)$ such that

$$l(x_0) = l(y) < \infty$$

since $l(x)$ and $l(y)$ are nonnegative integers. As X is caténaire, by *Claim 2.9* below we choose x_0 to be the point such that

$$l(x_0) = l(x_0, u).$$

It follows that we have

$$l(x, x_0) = l(x, u) - l(x_0, u) = l(x) - l(y) > 0.$$

Then $x_0 \neq y$. It is seen that x_0 and y are Sp -disconnected. Otherwise, if $x_0 \rightarrow y$ is a specialization, it is seen that $x \rightarrow y$ is a specialization, which will be in contradiction to the assumption in this subcase. If $y \rightarrow x_0$ is a specialization, we have

$$l(x_0) = l(y) = l(y, x_0) + l(x_0) \geq 1 + l(x_0)$$

since X is caténaire, where there will be a contradiction.

Thus, x_0 and y are Sp -disconnected and of the same length. As f is level-separated, we have

$$f(x_0) \neq f(y).$$

By *Claim 2.9* we have

$$l(\xi, x_0) = \dim X - l(x_0) = \dim X - l(y) = l(\xi, y);$$

then

$$l(\xi, y) = l(\xi, x_0) = l(\xi, x) + l(x, x_0).$$

As f is length-preserving, we have

$$l(f(x), f(x_0)) = l(x, x_0) > 0;$$

hence,

$$\begin{aligned} & l(f(\xi), f(y)) \\ &= l(\xi, y) \\ &= l(\xi, x_0) \\ &= l(f(\xi), f(x_0)) \\ &= l(f(\xi), f(x)) + l(f(x), f(x_0)) \\ &\leq l(Y) \\ &= \dim Y \\ &< \infty. \end{aligned}$$

We must have

$$f(x) \neq f(y).$$

Otherwise, if $f(x) = f(y)$, we have

$$\begin{aligned} & l(f(\xi), f(x)) \\ &= l(f(\xi), f(y)) \\ &= l(f(\xi), f(x)) + l(f(x), f(x_0)); \end{aligned}$$

then

$$l(x, x_0) = l(f(x), f(x_0)) = 0;$$

it follows that $x = x_0$ holds, where there will be a contradiction. Therefore, f is injective.

This completes the proof. \square

Claim 2.9. Let X be a caténaire and irreducible scheme of finite dimension. Let ξ be the generic point of X .

(i) We have

$$l(x_0, x_r) = l(x_0, x_1) + l(x_1, x_2) + \cdots + l(x_{r-1}, x_r)$$

for any chain of specializations $x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_r$ in X .

(ii) For any point x of X , we have

$$l(\xi, x) = \dim X - l(x).$$

In particular, we have

$$l(\xi, u) = \dim X$$

for any closed point u of X .

Proof. (i) It is immediate by induction on r . In fact, take any specializations

$$x \rightarrow y \rightarrow z$$

in X . We have irreducible closed subsets

$$Sp(x) \supseteq Sp(y) \supseteq Sp(z).$$

As X is a caténaire space^[1], we have

$$l(x, z) = l(x, y) + l(y, z).$$

(ii) Let u be a closed point of X . From the property of caténaire spaces^[1], we have

$$l(\xi, u) = l(X) = \dim X$$

by taking a presentation of specializations for the length $l(X)$.

By (i) it is easily seen that

$$l(\xi, x) = \dim X - l(x)$$

for any point x of X . \square

Corollary 2.10. Let $f : X \rightarrow Y$ be a morphism of schemes and let Y be of finite dimension. Then we have

$$\|f\| = 1$$

if f is injective and of Sp-type.

Proof. By *Theorem 2.7* we have $0 \leq \|f\| \leq 1$. As f is length-preserving by *Theorem 2.8*, we have

$$l(x, y) = l(f(x), f(y))$$

for any specialization $x \rightarrow y$ in X . Hence, $\|f\| = 1$ holds. \square

Remark 2.11. There are some concrete examples from commutative rings shows that the local condition, the Sp-type, can not be removed from the above theorems.

Remark 2.12. The topic on combinatorial norms of morphisms of schemes, discussed above in the paper, can be applied to study the discrete Morse theory on arithmetic schemes and Kontsevich's theory of graph homology.

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